A STOCHASTIC MODEL TO REPRESENT THE RATIO BETWEEN OIL PRICE AND GOLD PRICE IN USA

A.G.K.N.Alupotha*[1], A.A.I.Perera[2], P.M.L.K.Karunaratne[3]
[1] Department of Mathematics and Philosophy of Engineering, The Open University of Sri Lanka, Polgolla, Sri Lanka. (E-mail: alupotha@gmail.com)
[2] Department of Mathematics, University of Peradeniya, Peradeniya, Sri Lanka.
[3] Ceylon Petroleum Corporation, Colombo, Sri Lanka.

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Introduction

Perusal of much documentation on world economy suggests that the movements of oil price and gold price represent the behavior of the world's economy. Even though oil price is sensitive to world geo-political climate, it has a significant impact on world economy. Gold is seen as an inflation hedge and gold price tend to increase in slump economic conditions. Typically gold price has a positive correlation to oil price. Sometimes investors begin to view oil as an alternate investment in the periods of economic downturn. However, these scenarios have a significant impact on different economic conditions such as high inflationary conditions and social unrest. Hence, inspection of oil price and gold price will give us a clear illustration of the world economy. In USA, fluctuation of the ratio (oil price/gold price) has been a stochastic process and there is a mean reversion of the variation. Therefore, Vasicek model can be used to model the variation of aforementioned ratio. Vasicek model is the first economic model to capture the value of mean reversion and it is given by the following stochastic differential equation:

$$dr(t) = -\theta[r(t) - \mu]dt + \sigma dw\,; r(0) = r_0, \text{ where } r(t) = \frac{\text{oil price at time t}}{\text{gold price at time t}} \text{ and}$$

θ = Mean reversion speed

μ = Mean reversion parameter

σ = Standard deviation that determines the volatility of the ratio

w = Wiener process that models the risk factor of random market

Methodology

First and foremost, since 1991 to 2010 daily data pertaining to oil price (price of a barrel of light crude oil in US dollars) and gold price (price of one ounce in US dollars) were obtained from web sites of Yahoo Finance. Maximum likelihood estimators (MLE) for this data set were used to derive the values of Vasicek parameters θ, μ and σ as follows:

Consider $dr = -\theta(r - \mu)dt + \sigma dw\,; r(0) = r_0$.

Let $X = r - \mu$.

Then Vasicek model can be rewritten as $dX = -\theta X + \sigma dw$.

Define $y(t) = X(t)e^{\theta t}$.

Hence, we will get $dy = \sigma e^{\theta t} dw$ and this implies $y(t) = y(0) + \sigma \int_0^t e^{\sigma\tau} dw(\tau)$.

Therefore, $y(t) \sim Normal\left(X_0, \sigma^2 \int_0^t e^{2\theta\tau} d\tau\right)$ as $dw \sim Normal(0, d\tau)$.

This implies $X(t) \sim Normal\left(X_0 e^{-\theta t}, \sigma^2 e^{-2\theta t} \int_0^t e^{2\theta\tau} d\tau\right)$.

Therefore, we have $r(t) \sim Normal\left\{\mu + (r_0 - \mu)e^{-\theta t}, \sigma^2 e^{-2\theta t} \int_0^t e^{2\theta\tau} d\tau\right\}$.

i.e. $r(t) \sim Normal\left[\mu + (r_0 - \mu)e^{-\theta t}, \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)\right]$.

Then the probability density function of Vasicek model is given by

$$p(r) = \left[\frac{\pi\sigma^2}{\theta}\left(1 - e^{-2\theta t}\right)\right]^{\frac{-1}{2}} exp\left\{-\frac{\theta\left[r - \mu - (r_0 - \mu)e^{-t}\right]^2}{\sigma^2\left(1 - e^{-2\theta t}\right)}\right\}.$$

Then the Likelihood function is

$$L\left(\{\theta, \mu, \sigma^2\} \middle| r_0, r_1, \ldots, r_{n-1}\right) = p\left(r_0, r_1, \ldots, r_{n-1} \middle| \{\theta, \mu, \sigma^2\}\right) = \prod_{i=0}^{n-1} p\left(r_i \middle| \{\theta, \mu, \sigma^2\}\right)$$

Now define $L' = \frac{1}{n} lnL$.

Then

$$L' = \frac{1}{n} \sum_{i=0}^{n-1} ln\, p\left(r_i \middle| \{\theta, \mu, \sigma^2\}\right) =$$

$$\left(\frac{-1}{2}\right) \sum_{i=0}^{n-1} ln\left[\frac{\pi\sigma^2}{\theta}\left(1 - e^{-2\theta t}\right)\right]\left\{-\frac{\theta\left[r_i - \mu - (r_0 - \mu)e^{-\theta t}\right]^2}{\sigma^2\left(1 - e^{-2\theta t}\right)}\right\}.$$

It is natural to estimate the following funtions of the estimators:

$$\alpha := e^{-\theta t}, \qquad \beta := \mu, \qquad \gamma^2 := \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$$

Now consider the equations $\frac{\partial L'}{\partial \alpha} = 0, \frac{\partial L'}{\partial \beta} = 0, \frac{\partial L'}{\partial \gamma} = 0$ and evaluating these partial differential equations at $\alpha = \alpha', \beta = \beta'$ and $\gamma = \gamma'$ respectively, maximum likelihood estimators (MLE) for $\alpha, \beta,$ and γ^2 can be obtained as follows:

$$\alpha' = \frac{n\sum_{i=1}^{n} r_i r_{i-1} - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} r_{i-1}}{n\sum_{i=1}^{n} r_i^2 - \left(\sum_{i=1}^{n} r_{i-1}\right)^2}$$

$$\beta' = \frac{\sum_{i=1}^{n}[r_i - \alpha' r_{i-1}]}{n(1-\alpha')}$$

$$\gamma'^2 = \frac{1}{n}\sum_{i=1}^{n}[r_i - \alpha' r_{i-1} - \beta'(1-\alpha')]^2$$

Hence, stochastic differential equation for the ratio (oil price/gold price) can be derived.

Experimental results

According to the numerical values of α', β' and γ', values for Vasicek parameters are given by $\theta = 0.0037, \mu = 0.0791$ and $\sigma = 0.0023$. Therefore, final stochastic model for the ratio(oil price/gold price) is going to be

$$dr(t) = -0.0037[r(t) - 0.0791]dt + 0.0023dw, \text{ where } r(t) = \frac{\text{oil price at time t}}{\text{gold price at time t}}.$$

Therefore, simulation can be done using this derived stochastic model and Figure 1 will give us a clear illustration of actual values of the ratio and simulated values of the ratio.

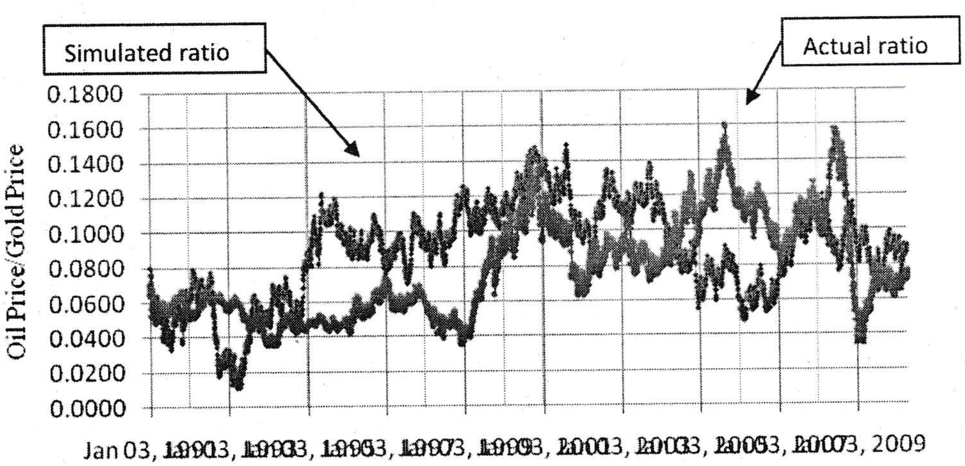

Figure 1

Discussion and Conclusion

Final stochastic model can definitely be used to forecast the ratio between oil price and gold price in USA and deterministic part of the final stochastic differential equation is going to be

$$dr(t) = -0.0037[r(t) - 0.0791]dt$$

. Therefore, deterministic process of the final stochastic model is given by

$$r(t) = 0.0791 - 0.0112exp(-0.0037t)$$

and hence this implies for long term periods r attains to 0.0791. Moreover, Monte Carlo simulation is the best strategy to minimize the gap between actual and simulated ratio in Figure 1.

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